Seat No. _________________ **Enrollment No._________________**

C. U. SHAH UNIVERSITY

M.Sc. (Mathematics) Semester-II Summer - 2015 Regular Examination

Subject Name: Number Theory **Subject Code: 5SC02MTE3**

Time: 03 hours **Maximum Marks: 70**

Instructions:

1. Attempt all questions.
2. Make suitable assumption whenever necessary.
3. Figures to the right indicate full marks.

Section – I

Q-1 a) Define: Prime number. If p is prime and $p|ab$, then prove that $p|a$ or $p|b$. (02)

b) If $a|bc$ and$(a,b) = 1$, then prove that $a|c$. (02)

c) Findgcd$(306, 657)$. (02)

d) Prove that square of any odd number is of the form $8k + 1, k$ is integer. (01)

Q-2 a) Define: Greatest common divisor. If$(a,b) = d$, then prove that there exists x and y such that $ax + by = d$. (05)

b) Define: Mobious function. Prove that Mobious function is multiplicative function. (05)

c) Prove that $\tau(n)$ is an odd integer if and only if n is a perfect square. (04)

OR

Q-2 a) State and prove fundamental theorem of divisibility. (05)

b) Define: Multiplicative function. If f is a multiplicative function and F is defined by$F(n) = \sum_{d|n} f(d)$, then prove that F is also multiplicative. (05)

c) If $p_n$is the n^{th}prime numbers, then prove that $p_n \le 2^{2^{n-1}}, \forall\, n$. (04)

Q-3 a) State and prove relation between LCM and GCD of two number. (05)

b) Solve the system of three congruences (05)

$$x \equiv 1(mod\ 4), x \equiv 3(mod\ 5), x \equiv 2(mod\ 7)$$

c) If $p_n$is the n^{th}prime numbers, then prove that $p_n \le 2^{2^{n-1}}, \forall\, n$. (04)

OR

Q-3 a) State and prove Chinese Remainder theorem. (05)

b) Prove that if p is a prime and (05)

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 a_n \not\equiv 0\ (mod\ p)$$

is a polynomial of degree $n \ge 1$ with integral coefficients, then the congruence $f(x) \equiv 0(mod\ p)$ has at most n incongruent solution modulo p.

c) Prove that the function τ and σ are both multiplicative functions. (04)

Section – II

Q-4 a) Determine the infinite continued fraction representation of$\sqrt{5}$. (02)

b) If the integer a has order k modulo n, then prove that $a^i \equiv a^j (mod\ n)$ if and only if $i \equiv j(mod\ n)$. (02)

c) Define: Euler phi-function. Also find $\phi(9)$. (02)

d) The value of any infinite continue fraction is an irrational number. Determine whether the statement is true or false. (01)

Q-5 a) Let $N = a_0 + a_1 10 + a_2 10^2 + \cdots + a_m 10^m$ be the decimal expansion of the positive integer $N, 0 \leq a_k < 10$, and let $S = a_0 + a_1 + \cdots + a_m$. Then prove that $9|N$ if and only if $9|S$. Is 1571724 divisible by 9? Justify. (05)

b) Prove that the product of two primitive polynomial is primitive. (05)

c) Find all integer solutions of the equation $56x + 72y = 40$. (04)

OR

Q-5 a) Define: Infinite continued fractions. Find the unique irrational number represented by the infinite continued fraction $[1; 2, \overline{3, 1}]$. (05)

b) If $\frac{p_k}{q_k}$ arethe convergents of the continued fraction expansion of $\sqrt{d}$then, prove that $p_k^2 - dq_k^2 = (-1)^{k+1} t_{k+1}$, where $t_{k+1} > 0, k = 0,1,2,3, \ldots$ (05)

c) Prove that for any integer $n > 1, \phi(n) = n - 1$ if and only if n is prime. (04)

Q-6 a) Prove that all the solutions of $x^2 + y^2 = z^2$with $x, y, z > 0$; satisfying the conditions $(x, y, z) = 1, \ 2|x$are given by the formula $x = 2st, y = s^2 - t^2, \ z = s^2 + t^2$,where$s > t > 0, (s, t) = 1$ and one of s, t is even and the other is odd. (05)

b) Prove that the product of two primitive polynomial is primitive. (05)

c) Compute the convergents of the simple continued fraction $[1; 2,3,3,2,1]$. (04)

OR

Q-6 a) If $2^k - 1$ is prime$(k > 1)$, then prove that $n = 2^{k-1}(2^k - 1)$ is perfect and every even perfect number is of the form. (05)

b) Prove that any rational number can be written as a finite simple continued fraction. (05)

c) Prove that if an irreducible polynomial $p(x)$ divides a product $f(x)g(x)$, then $p(x)$ divides at least one of the polynomials $f(x)$ and $g(x)$. (04)

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